

AC-CONDUCTIVITY AND ELECTROMAGNETIC ENERGY ABSORPTION FOR THE ANDERSON MODEL IN LINEAR RESPONSE THEORY

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Dedicated to Leonid A. Pastur on the occasion of his 75th birthday

ABSTRACT. We continue our study of the ac-conductivity in linear response theory for the Anderson model using the conductivity measure. We establish further properties of the conductivity measure, including nontriviality at nonzero temperature, the high temperature limit, and asymptotics with respect to the disorder. We also calculate the electromagnetic energy absorption in linear response theory in terms of the conductivity measure.

CONTENTS

1. Introduction	1
2. Nontriviality and high temperature limit of the conductivity measure	6
3. Asymptotics with respect to the disorder	8
4. Electromagnetic energy absorption	11
References	12

1. INTRODUCTION

We continue our study of the ac-conductivity in linear response theory for the Anderson model following [KILM, KIM], where we introduced the concept of a conductivity measure. The conductivity measure $\Sigma_\mu^T(d\nu)$ at absolute temperature $T \geq 0$ and Fermi level (chemical potential) $\mu \in \mathbb{R}$ is a finite positive even Borel measure on frequency space (ν denotes the frequency of the applied electric field). If $\Sigma_\mu^T(d\nu)$ was known to be an absolutely continuous measure, the in-phase (or active) conductivity $\operatorname{Re} \sigma_\mu^T(\nu)$ would then be well-defined as its density. The conductivity measure $\Sigma_\mu^T(d\nu)$ is an analogous concept to the density of states measure, whose formal density is the density of states. The Mott formula proved in [KILM] is a statement about the asymptotic behavior of $\Sigma_\mu^0([0, \nu])$ as $\nu \downarrow 0$ for a Fermi level μ within the region of complete localization.

In this article we establish further properties of the conductivity measure, including nontriviality at nonzero temperature, the high temperature limit, and asymptotics with respect to the disorder. We also calculate the electromagnetic energy absorption in linear response theory in terms of the conductivity measure.

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This work is motivated by a series of papers by Bru, de Siqueira Pedra and Kurig* [BrPK1, BrPK2, BrPK3, BrPK4], who study the linear response of an infinite system of free fermions in the lattice to an electric field given by a time and space dependent potential. They assume the presence of impurities, and take the one-particle Hamiltonian to be the Anderson model. They show that at nonzero temperature ($T > 0$) the electromagnetic energy absorbed (‘heat production’) in linear response theory is given in terms of an ac-conductivity measure. They also derive asymptotics for this ac-conductivity measure with respect to the disorder. Although the BPK (for Bru, de Siqueira Pedra and Kurig) mathematical setting is very different from ours (see, e.g., the discussion in [BrPK3, Section 2.3]), there are clear similarities between their work and ours.

The Anderson model is described by the random Schrödinger operator H , a measurable map $\omega \mapsto H_\omega$ from a probability space (Ω, \mathbb{P}) (with expectation \mathbb{E}) to bounded self-adjoint operators on $\ell^2(\mathbb{Z}^d)$, given by

$$H_\omega := -\Delta + V_\omega, \quad (1.1)$$

where Δ is the centered discrete Laplacian

$$(\Delta\varphi)(x) := - \sum_{y \in \mathbb{Z}^d; |x-y|=1} \varphi(y) \quad \text{for } \varphi \in \ell^2(\mathbb{Z}^d), \quad x \in \mathbb{Z}^d, \quad (1.2)$$

and the random potential V_ω consists of independent, identically distributed random variables $\{V_\omega(x); x \in \mathbb{Z}^d\}$ on (Ω, \mathbb{P}) , such that the common single site probability distribution ξ is nondegenerate with compact support, say

$$\{v_-, v_+\} \in \text{supp } \xi \subset [v_-, v_+], \quad \text{where } -\infty < v_- < v_+ < \infty, \quad (1.3)$$

and has a bounded density $\rho \in L^\infty(\mathbb{R})$.

The Anderson Hamiltonian H given by (1.1) is \mathbb{Z}^d -ergodic. It follows that its spectrum is nonrandom: there exists a set $\mathfrak{S} \subset \mathbb{R}$ such that $\sigma(H_\omega) = \mathfrak{S}$ with probability one [P3, P4]. Moreover, the pure point, absolutely continuous, and singular continuous components of $\sigma(H_\omega)$ are also nonrandom, i.e., equal to fixed sets $\mathfrak{S}_{\text{pp}}, \mathfrak{S}_{\text{ac}}, \mathfrak{S}_{\text{sc}}$ with probability one, and $\mathfrak{S} = [-2d, 2d] + \text{supp } \xi$ (see [KiM, CL, PF, Ki]). In particular, setting $E_- := -2d + v_-$ and $E_+ := 2d + v_+$, we have

$$-\infty < E_- = \inf \mathfrak{S} < E_+ = \sup \mathfrak{S} < \infty, \quad \text{so } \{E_-, E_+\} \subset \mathfrak{S} \subset [E_-, E_+]. \quad (1.4)$$

We start by reviewing the derivation of electrical ac-conductivities within linear response theory for the Anderson model following [BoGKS, KILM, KIM]. At time $t = -\infty$, the system is in thermal equilibrium at absolute temperature $T \geq 0$ and chemical potential $\mu \in \mathbb{R}$. In the single-particle Hilbert space $\ell^2(\mathbb{Z}^d)$ this equilibrium state is given by the random operator $f_\mu^T(H)$, where

$$f_\mu^T(E) := \begin{cases} \left(e^{\frac{E-\mu}{T}} + 1\right)^{-1} & \text{if } T > 0 \\ \chi_{]-\infty, \mu]}(E) & \text{if } T = 0 \end{cases} \quad (1.5)$$

is the Fermi function. By χ_B we denote the characteristic function of the set B . A spatially homogeneous, time-dependent electric field $\mathbf{E}(t)$ is then introduced adiabatically: Starting at time $t = -\infty$, we switch on the (adiabatic) electric field $\mathbf{E}_\eta(t) := e^{\eta t} \mathbf{E}(t)$ with $\eta > 0$, and then let $\eta \rightarrow 0$.

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In view of isotropy we assume without loss of generality that the electric field points in the x_1 -direction: $\mathbf{E}(t) = \mathcal{E}(t)\hat{x}_1$, where $\mathcal{E}(t)$ is the (real-valued) amplitude of the electric field, and \hat{x}_1 is the unit vector in the x_1 -direction. We assume

$$\mathcal{E}(t) = \int_{\mathbb{R}} d\nu e^{i\nu t} \hat{\mathcal{E}}(\nu), \quad \text{where } \hat{\mathcal{E}} \in C(\mathbb{R}) \cap L^1(\mathbb{R}) \text{ with } \hat{\mathcal{E}}(\nu) = \overline{\hat{\mathcal{E}}(-\nu)}. \quad (1.6)$$

Note that (1.6) implies that $\mathcal{E} \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and is real-valued.

For each $\eta > 0$ this procedure results in a time-dependent random Hamiltonian

$$H_\omega(\eta, t) := G(\eta, t) H_\omega G(\eta, t)^*, \quad \text{with } G(\eta, t) := e^{iX_1 \int_{-\infty}^t ds e^{\eta s} \mathcal{E}(s)}, \quad (1.7)$$

where X_1 stands for the operator of multiplication by the first coordinate of the electron's position. (The time-dependent, bounded Hamiltonian $H_\omega(\eta, t)$ is gauge equivalent to $H_\omega + e^{\eta t} \mathcal{E}(t) X_1$; this choice of gauge is discussed in [BoGKS, Section 2.2].) The state of the system is described at time t by the random operator $\varrho_{\mu, \omega}^T(\eta, t)$, the solution to the Liouville equation

$$\begin{cases} i\partial_t \varrho_{\mu, \omega}^T(\eta, t) = [H_\omega(\eta, t), \varrho_{\mu, \omega}^T(\eta, t)] \\ \lim_{t \rightarrow -\infty} \varrho_{\mu, \omega}^T(\eta, t) = f_\mu^T(H_\omega) \end{cases}. \quad (1.8)$$

The adiabatic electric field generates a time-dependent electric current, also oriented along the first coordinate axis. In the Schrödinger picture it has amplitude

$$J_\eta(t; \mu, T, \mathcal{E}) = -\mathcal{T}(\varrho_{\mu, \omega}^T(\eta, t) \dot{X}_1(\eta, t)), \quad (1.9)$$

where \mathcal{T} is the trace per unit volume, defined below Eq. (1.13), and $\dot{X}_1(\eta, t)$ is the first component of the time-dependent velocity operator:

$$\dot{X}_1(\eta, t) := G(\eta, t) \dot{X}_1 G(\eta, t)^*, \quad \text{where } \dot{X}_1 := i[H_\omega, X_1] = i[-\Delta, X_1]. \quad (1.10)$$

The *adiabatic linear-response current* is defined as

$$J_{\eta, \text{lin}}(t; \mu, T, \mathcal{E}) := \frac{d}{d\alpha} J_\eta(t; \mu, T, \alpha \mathcal{E})|_{\alpha=0}. \quad (1.11)$$

The detailed analysis in [BoGKS] gives a mathematical meaning to the formal procedure leading to (1.11), for fixed temperature $T \geq 0$ and chemical potential $\mu \in \mathbb{R}$, when the corresponding thermal equilibrium random operator $f_\mu^T(H_\omega)$ satisfies the condition

$$\mathbb{E}\{\|X_1 f_\mu^T(H_\omega) \delta_0\|^2\} < \infty, \quad (1.12)$$

where $\{\delta_a\}_{a \in \mathbb{Z}^d}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}^d)$: $\delta_a(x) := 1$ if $x = a$ and $\delta_a(x) := 0$ otherwise. (This condition appears in [BES].) This analysis requires the mathematical framework of normed spaces of measurable covariant operators given in [BoGKS, Section 3], and described in [KILM, Section 3] and [KIM, Appendix A]. Here we will only give a short (and informal) review. By \mathcal{K}_2 we denote the Hilbert space of measurable covariant operators A on $\ell^2(\mathbb{Z}^d)$, i.e., measurable, covariant maps $\omega \mapsto A_\omega$ from the probability space (Ω, \mathbb{P}) to operators on $\ell^2(\mathbb{Z}^d)$, with inner product

$$\langle\langle A, B \rangle\rangle := \mathbb{E}\{\langle A_\omega \delta_0, B_\omega \delta_0 \rangle\} = \mathcal{T}\{A^* B\} \quad (1.13)$$

and norm $\|A\|_2 := \sqrt{\langle\langle A, A \rangle\rangle}$. Here \mathcal{T} , given by $\mathcal{T}(A) := \mathbb{E}\{\langle \delta_0, A_\omega \delta_0 \rangle\}$, is the trace per unit volume. The Liouvillian \mathcal{L} is the (bounded in the case of the Anderson model) self-adjoint operator on \mathcal{K}_2 given by the commutator with H :

$$(\mathcal{L}A)_\omega := [H_\omega, A_\omega]. \quad (1.14)$$

We also introduce the operators \mathcal{H}_L and \mathcal{H}_R on \mathcal{K}_2 that are given by left and right multiplication by H :

$$(\mathcal{H}_L A)_\omega := H_\omega A_\omega \quad \text{and} \quad (\mathcal{H}_R A)_\omega := A_\omega H_\omega. \quad (1.15)$$

They are commuting, bounded (for the Anderson Hamiltonian), self-adjoint operators on \mathcal{K}_2 , anti-unitarily equivalent, and $\mathcal{L} = \mathcal{H}_L - \mathcal{H}_R$. It is easy to see that

$$\sigma(\mathcal{H}_L) = \sigma(\mathcal{H}_R) \subset \mathfrak{S}, \quad \text{so} \quad \sigma(\mathcal{L}) = \sigma(\mathcal{H}_L) - \sigma(\mathcal{H}_R) \subset \mathfrak{S} - \mathfrak{S}. \quad (1.16)$$

It follows from the Wegner estimate for the Anderson model that the operators \mathcal{H}_L and \mathcal{H}_R have purely absolutely continuous spectrum [KIM, Lemma 1]. For each $T \geq 0$ and $\mu \in \mathbb{R}$ we consider the bounded self-adjoint operator \mathcal{F}_μ^T in \mathcal{K}_2 given by

$$\mathcal{F}_\mu^T := f_\mu^T(\mathcal{H}_L) - f_\mu^T(\mathcal{H}_R), \quad \text{i.e.,} \quad (\mathcal{F}_\mu^T A)_\omega = [f_\mu^T(H_\omega), A_\omega]. \quad (1.17)$$

In this formalism the condition (1.12) can be rewritten as

$$Y_\mu^T := i[X_1, f_\mu^T(H)] \in \mathcal{K}_2, \quad (1.18)$$

which is always true for $T > 0$ and arbitrary $\mu \in \mathbb{R}$, since in this case $f_\mu^T(H) = g(H)$ for some Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ ([BoGKS, Remark 5.2(iii)]). We set

$$\Xi_0 := \{\mu \in \mathbb{R}; \quad Y_\mu^0 \in \mathcal{K}_2\}. \quad (1.19)$$

For the same reason as when $T > 0$, we have $\mu \in \Xi_0$ if either $\mu \notin \mathfrak{S}$ or μ is the left edge of a spectral gap for H . Moreover, letting Ξ^{cl} denote the region of complete localization (see [GK]), defined as the region of validity of the multiscale analysis, or equivalently, of the fractional moment method, we have (see [AG, GK])

$$\Xi^{\text{cl}} \subset \Xi_0. \quad (1.20)$$

We refer to [KIM, Appendix B] for a precise definition. Note $\mathbb{R} \setminus \mathfrak{S} \subset \Xi^{\text{cl}}$ and that Ξ^{cl} is an open set by its definition. Note also that for $\mu \in \Xi^{\text{cl}}$ the Fermi projection $f_\mu^0(H)$ satisfies a much stronger condition than (1.12), namely exponential decay of its kernel [AG, Theorem 2]. Conversely, fast enough polynomial decay of the kernel of the Fermi projection for all energies in an interval implies complete localization in the interval [GK, Theorem 3].

If $Y_\mu^T \in \mathcal{K}_2$, an inspection of the proof of [BoGKS, Thm. 5.9] shows that the adiabatic linear-response current (1.11) is well defined for all $t \in \mathbb{R}$, and given by (see [KIM, Eq. (2.18)])

$$\begin{aligned} J_{\eta, \text{lin}}(t; \mu, T, \mathcal{E}) &= \mathcal{T} \left\{ \int_{-\infty}^t ds \, e^{\eta s} \mathcal{E}(s) \dot{X}_1 e^{-i(t-s)\mathcal{L}} Y_\mu^T \right\} \\ &= \int_{-\infty}^t ds \, e^{\eta s} \mathcal{E}(s) \langle\langle \dot{X}_1, e^{-i(t-s)\mathcal{L}} Y_\mu^T \rangle\rangle. \end{aligned} \quad (1.21)$$

We introduced the conductivity measure in [KILM, KIM] to rewrite (1.21). If $Y_\mu^T \in \mathcal{K}_2$, the (ac-)conductivity measure (x_1 - x_1 component) at temperature T and Fermi level μ is defined by

$$\Sigma_\mu^T(B) := \pi \langle\langle \dot{X}_1, \chi_B(\mathcal{L}) Y_\mu^T \rangle\rangle \quad \text{for all Borel sets } B \subset \mathbb{R}. \quad (1.22)$$

We proved that Σ_μ^T is a finite positive even Borel measure on the real line [KIM, Theorem 1]. Thus (1.21) can be rewritten as

$$J_{\eta,\text{lin}}(t; \mu, T, \mathcal{E}) = \frac{1}{\pi} \int_{-\infty}^t ds e^{\eta s} \mathcal{E}(s) \int_{\mathbb{R}} \Sigma_\mu^T(d\lambda) e^{-i(t-s)\lambda} \quad (1.23)$$

$$= e^{\eta t} \int_{\mathbb{R}} d\nu e^{i\nu t} \sigma_\mu^T(\eta, \nu) \widehat{\mathcal{E}}(\nu), \quad (1.24)$$

where

$$\sigma_\mu^T(\eta, \nu) := -\frac{i}{\pi} \int_{\mathbb{R}} \Sigma_\mu^T(d\lambda) \frac{1}{\lambda + \nu - i\eta}. \quad (1.25)$$

We then defined the adiabatic in-phase linear-response current by

$$J_{\eta,\text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) := e^{\eta t} \int_{\mathbb{R}} d\nu e^{i\nu t} (\text{Re } \sigma_\mu^T(\eta, \nu)) \widehat{\mathcal{E}}(\nu). \quad (1.26)$$

Turning off the adiabatic switching, we obtained a simple expression for the in-phase linear-response current in terms of the conductivity measure, given by

$$J_{\text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) := \lim_{\eta \downarrow 0} J_{\eta,\text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) = \int_{\mathbb{R}} \Sigma_\mu^T(d\nu) e^{i\nu t} \widehat{\mathcal{E}}(\nu). \quad (1.27)$$

This derivation of the in-phase linear-response current is valid as long as either $T > 0$ or $\mu \in \Xi_0$, so we can guarantee (1.18). In addition, we proved [KIM, Eq. (2.31)] that

$$J_{\text{lin}}^{\text{in}}(t; \mu, 0, \mathcal{E}) = \lim_{T \downarrow 0} J_{\text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) \quad \text{for all } \mu \in \Xi_0. \quad (1.28)$$

In [KIM] we also extended the definition of the conductivity measure at $T = 0$ to arbitrary Fermi level μ . Given $T > 0$ and $\mu \in \mathbb{R}$, we decompose Σ_μ^T as

$$\Sigma_\mu^T = \Sigma_\mu^T(\{0\}) \delta_0 + (\Sigma_\mu^T - \Sigma_\mu^T(\{0\}) \delta_0) = \Psi((-f_\mu^T)') \delta_0 + \Gamma_\mu^T, \quad (1.29)$$

where δ_0 is the Dirac measure at 0, and Ψ and Γ_μ^T are finite positive Borel measures on \mathbb{R} given by

$$\Psi(B) := \pi \langle \dot{X}_1, \chi_{\{0\}}(\mathcal{L}) \chi_B(\mathcal{H}_L) \dot{X}_1 \rangle, \quad (1.30)$$

$$\Gamma_\mu^T(B) := \pi \langle (-\mathcal{L}_\perp^{-1} \mathcal{F}_\mu^T)^{\frac{1}{2}} \dot{X}_1, \chi_B(\mathcal{L}) (-\mathcal{L}_\perp^{-1} \mathcal{F}_\mu^T)^{\frac{1}{2}} \dot{X}_1 \rangle. \quad (1.31)$$

Here \mathcal{L}_\perp^{-1} denotes the pseudo-inverse of \mathcal{L} (i.e., $\mathcal{L}_\perp^{-1} := g(\mathcal{L})$ where $g(t) := \frac{1}{t}$ if $t \neq 0$ and $g(0) := 0$) and \mathcal{F}_μ^T is given in (1.17). Note that $\Gamma_\mu^T(\{0\}) = 0$. (In (1.29) we used the short-hand notation $\Phi(h) := \int_{\mathbb{R}} \Phi(d\lambda) h(\lambda)$ for the integral of a function h with respect to the measure Φ .)

We let $\mathcal{M}(\mathbb{R})$ denote the vector space of complex Borel measures on \mathbb{R} , with $\mathcal{M}_+(\mathbb{R})$ being the cone of finite positive Borel measures, and with $\mathcal{M}_+^{(e)}(\mathbb{R})$ the finite positive even Borel measures. We recall that $\mathcal{M}(\mathbb{R}) = C_0(\mathbb{R})^*$, where $C_0(\mathbb{R})$ denotes the Banach space of complex-valued continuous functions on \mathbb{R} vanishing at infinity with the sup norm. We will use three locally convex topologies on $\mathcal{M}(\mathbb{R})$. The first is the weak* topology (also called the vague topology), induced by the linear functionals $\{\Gamma \in \mathcal{M}(\mathbb{R}) \mapsto \Gamma(g); g \in C_0(\mathbb{R})\}$. The second is the weak topology, defined in the same way as the weak* topology but with $C_b(\mathbb{R})$, the bounded continuous functions on \mathbb{R} , substituted for $C_0(\mathbb{R})$. The third is the strong topology, induced by the linear functionals $\{\Gamma \in \mathcal{M}(\mathbb{R}) \mapsto \Gamma(B); B \subset \mathbb{R} \text{ Borel set}\}$. We will write w*-lim, w-lim, and s-lim, to denote limits in the weak*, weak, and strong topology, respectively.

We proved [KIM, Theorem 2] that the measure Ψ from (1.30) is absolutely continuous with density ψ , where $\psi(E) = 0$ on Ξ_0 since $\text{supp } \Psi \subset \overline{\mathbb{R} \setminus \Xi_0} \subset \mathbb{R} \setminus \Xi_0^{\text{cl}}$. We defined

$$\Sigma_\mu^0 := \psi(\mu)\delta_0 + \Gamma_\mu^0, \quad (1.32)$$

which coincides with the previous definition for $\mu \in \Xi_0$, and yields

$$\Sigma_\mu^0(d\nu) = \text{w}^*\text{-}\lim_{T \downarrow 0} \Sigma_\mu^T(d\nu) \quad \text{for Lebesgue-a.e. } \mu \in \mathbb{R}. \quad (1.33)$$

Moreover, for all for all temperatures $T > 0$ and and Fermi levels $\mu \in \mathbb{R}$ we have

$$\Sigma_\mu^T = \left((-f_0^T)' * \Sigma_\bullet^0 \right)(\mu), \quad \text{i.e.,} \quad \Sigma_\mu^T(B) = \int_{\mathbb{R}} dE (-f_\mu^T)'(E) \Sigma_E^0(B), \quad (1.34)$$

which justifies the extension of the definition of in-phase linear-response current by

$$J_{\text{lin}}^{\text{in}}(t; \mu, 0, \mathcal{E}) := \int_{\mathbb{R}} \Sigma_\mu^0(d\nu) e^{i\nu t} \widehat{\mathcal{E}}(\nu) = \lim_{T \downarrow 0} J_{\text{lin}}^{\text{in}}(t; \mu, T, \mathcal{E}) \quad \text{for Lebesgue-a.e. } \mu \in \mathbb{R}. \quad (1.35)$$

(We refer to [KIM] for full details and proofs.)

In Section 2 we establish nontriviality of the conductivity measure at $T > 0$ and prove that $\Sigma_\mu^T \rightarrow 0$ strongly as $T \rightarrow \infty$. In Section 3 we introduce a disorder parameter λ and study the weak* limit of $\Sigma_{\mu,\lambda}^T$ as $\lambda \rightarrow 0$ and the strong limit of $\Sigma_{\mu,\lambda}^T$ as $\lambda \rightarrow \infty$. Finally, in Section 4 we derive an expression, in linear response theory, for the total electromagnetic energy $W_{\mu,\text{lin}}^T(\mathcal{E})$ absorbed by the solid during all times in terms of the conductivity measure Σ_μ^T :

$$W_{\mu,\text{lin}}^T(\mathcal{E}) = 2\pi \int_{\mathbb{R}} \Sigma_\mu^T(d\nu) |\widehat{\mathcal{E}}(\nu)|^2 \geq 0. \quad (1.36)$$

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2. NONTRIVIALITY AND HIGH TEMPERATURE LIMIT OF THE CONDUCTIVITY MEASURE

We prove nontriviality of the conductivity measure for $T > 0$ and strong convergence to 0 as $T \rightarrow \infty$. In the BPK setting, nontriviality of the ac-conductivity measure for sufficiently large T is shown in [BrPK4, Theorem 4.7].

Theorem 2.1. *Let $T > 0$ and $\mu \in \mathbb{R}$. Then the conductivity measure $\Sigma_{\mu,\lambda}^T$ and the measure $\Gamma_{\mu,\lambda}^T$ are nontrivial:*

$$\Sigma_\mu^T(\mathbb{R}) \geq \Gamma_\mu^T(\mathbb{R}) > 0. \quad (2.1)$$

Moreover, we have

$$\text{s-}\lim_{T \rightarrow \infty} \Sigma_\mu^T = \text{s-}\lim_{T \rightarrow \infty} \Gamma_\mu^T = 0 \quad \text{for all } \mu \in \mathbb{R}. \quad (2.2)$$

Remark 2.2. *It follows from (1.22) and (1.31) that*

$$\Sigma_\mu^T(\mathbb{R} \setminus \sigma(\mathcal{L})) = 0 \quad \text{and} \quad \Gamma_\mu^T\left((\mathbb{R} \setminus \sigma(\mathcal{L})) \cup \{0\}\right) = 0. \quad (2.3)$$

Thus only frequencies $\nu \in \sigma(\mathcal{L}) \subset \mathfrak{S} - \mathfrak{S}$ (recall (1.16)) contribute to Σ_μ^T and Γ_μ^T . It follows from Theorem 2.1 that

$$\Sigma_\mu^T(\sigma(\mathcal{L})) \geq \Gamma_\mu^T(\sigma(\mathcal{L}) \setminus \{0\}) > 0 \quad \text{for all } T > 0 \quad \text{and } \mu \in \mathbb{R}. \quad (2.4)$$

To prove the theorem we introduce the finite Borel measure Υ on \mathbb{R} given by

$$\Upsilon(B) := \langle \dot{X}_1, \chi_{B \setminus \{0\}}(\mathcal{L}) \dot{X}_1 \rangle \quad \text{for all Borel sets } B \subset \mathbb{R}. \quad (2.5)$$

Since $[\dot{X}_1, H_\omega] = [\dot{X}_1, V_\omega] \neq 0$ for a.e. ω , we have $\mathcal{L} \dot{X}_1 \neq 0$, so $\chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}) \dot{X}_1 \neq 0$. It follows that the measure Υ is nontrivial:

$$\Upsilon(\mathbb{R}) = \Upsilon(\mathbb{R} \setminus \{0\}) = \|\chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}) \dot{X}_1\|_2^2 > 0. \quad (2.6)$$

Lemma 2.3. *Let $T > 0$ and $\mu \in \mathbb{R}$. Then*

$$\frac{1}{4T} C_\mu^T \chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}) \leq -\mathcal{L}_\perp^{-1} \mathcal{F}_\mu^T \leq \frac{1}{4T} \chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}), \quad (2.7)$$

where

$$C_\mu^T := \inf_{E \in [E_- - E_+, E_+ - E_-]} \operatorname{sech}^2\left(\frac{E - \mu}{T}\right) > 0, \quad (2.8)$$

and E_\pm are as in (1.4). It follows that

$$\frac{\pi}{4T} C_\mu^T \Upsilon(B) \leq \Gamma_\mu^T(B) \leq \frac{\pi}{4T} \Upsilon(B) \quad \text{for all Borel sets } B \subset \mathbb{R}, \quad (2.9)$$

and the measure Γ_μ^T is nontrivial: $\Gamma_\mu^T(\mathbb{R}) > 0$.

As a consequence, we have

$$\lim_{T \rightarrow \infty} \Gamma_\mu^T = 0 \quad \text{for all } \mu \in \mathbb{R}. \quad (2.10)$$

Proof. Since (2.9) follows from (2.7) using [KIM, Eqs. (2.43)] and (2.5), it suffices to prove (2.7).

It follows from [KIM, Eqs. (2.39)–(2.40)] that for $T > 0$ we have

$$\begin{aligned} -\mathcal{L}_\perp^{-1} \mathcal{F}_\mu^T &= F_\mu^T(\mathcal{H}_L, \mathcal{H}_R) \leq \left(\sup_{(\lambda_1, \lambda_2) \in \mathbb{R}^2} F_\mu^T(\lambda_1, \lambda_2) \right) \chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}) \\ &\leq \left(\sup_{E \in \mathbb{R}} (-f_\mu^T(E))' \right) \chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}) = \frac{1}{4T} \chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}), \end{aligned} \quad (2.11)$$

where we used the mean-value theorem.

The lower bound is proved in a similar way. We have

$$\begin{aligned} -\mathcal{L}_\perp^{-1} \mathcal{F}_\mu^T &= F_\mu^T(\mathcal{H}_L, \mathcal{H}_R) \geq \left(\inf_{(\lambda_1, \lambda_2) \in [E_-, E_+]^2} F_\mu^T(\lambda_1, \lambda_2) \right) \chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}) \\ &\geq \left(\inf_{E \in [E_- - E_+, E_+ - E_-]} (-f_\mu^T(E))' \right) \chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}) \geq \widetilde{C}_\mu^T \chi_{\mathbb{R} \setminus \{0\}}(\mathcal{L}), \end{aligned} \quad (2.12)$$

where

$$\widetilde{C}_\mu^T := \frac{1}{T} \inf_{E \in [E_- - E_+, E_+ - E_-]} \frac{e^{\frac{E - \mu}{T}}}{(e^{\frac{E - \mu}{T}} + 1)^2} = \frac{1}{4T} C_\mu^T, \quad (2.13)$$

with C_μ^T as in (2.8). \square

Proof of Theorem 2.1. It follows from [KIM, Eqs. (2.41)] and (2.11) that

$$\Psi \left((-f_\mu^T)' \right) \leq \pi \|\chi_{\{0\}}(\mathcal{L}) \dot{X}_1\|_2^2 \sup_{E \in \mathbb{R}} (-f_\mu^T(E))' \leq \frac{\pi}{4T} \|\chi_{\{0\}}(\mathcal{L}) \dot{X}_1\|_2^2, \quad (2.14)$$

so

$$\lim_{T \rightarrow \infty} \Psi \left((-f_\mu^T)' \right) = 0 \quad \text{for all } \mu \in \mathbb{R}. \quad (2.15)$$

The theorem follows from (1.29), Lemma 2.3, and (2.15). \square

3. ASYMPTOTICS WITH RESPECT TO THE DISORDER

We now introduce a disorder parameter $\lambda \geq 0$. We consider the Anderson model H_λ , given by $H_{\omega,\lambda} := -\Delta + \lambda V_\omega$ (see (1.1)), and attach the label λ to all quantities considered, when appropriate. (Note that $H_0 = -\Delta$.) We consider the small and large disorder limits of the conductivity measure. Results of a similar nature in the BPK setting are given in [BrPK4, Theorem 4.6].

Theorem 3.1 (Small disorder). *For all $T > 0$ and $\mu \in \mathbb{R}$ we have*

$$\text{w-lim}_{\lambda \rightarrow 0} \Sigma_{\mu,\lambda}^T = \Sigma_{\mu,0}^T(\{0\}) \delta_0, \quad (3.1)$$

where

$$\Sigma_{\mu,0}^T(\{0\}) = \int_{\mathbb{R}} d\zeta (-f_\mu^T)'(\zeta) \Sigma_{\zeta,0}^0(\{0\}) = \int_{\mathbb{R}} d\zeta (-f_\mu^T)'(\zeta) \psi(\zeta) > 0. \quad (3.2)$$

Proof. To prove (3.1) it suffices to show convergence of the Fourier transforms for every $t \in \mathbb{R}$. Given $T > 0$ and $\mu \in \mathbb{R}$, we have

$$\begin{aligned} \int_{\mathbb{R}} \Sigma_{\mu,\lambda}^T(d\nu) e^{it\nu} &= \pi \langle\langle e^{it\mathcal{L}_\lambda} \dot{X}_1, Y_\mu^T \rangle\rangle = i\pi \langle\langle e^{it\mathcal{L}_\lambda} \dot{X}_1, [X_1, f_\mu^T(H_\lambda)] \rangle\rangle \\ &= i\mathbb{E} \left\{ \langle e^{itH_{\omega,\lambda}} \dot{X}_1 e^{-itH_{\omega,\lambda}} \delta_0, X_1 f_\mu^T(H_{\omega,\lambda}) \delta_0 \rangle \right\}. \end{aligned} \quad (3.3)$$

Since in view of (1.3) we have

$$\|H_{\omega,\lambda} - (-\Delta)\| = \|\lambda V_\omega\| \leq \lambda \max\{|v_-|, |v_+|\} \quad \text{with probability one,} \quad (3.4)$$

we get

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \int_{\mathbb{R}} \Sigma_{\mu,\lambda}^T(d\nu) e^{it\nu} &= i\mathbb{E} \left\{ \langle e^{it(-\Delta)} \dot{X}_1 e^{-it(-\Delta)} \delta_0, X_1 f_\mu^T(-\Delta) \delta_0 \rangle \right\} \\ &= \int_{\mathbb{R}} \Sigma_{\mu,0}^T(d\nu) e^{it\nu} = \Sigma_{\mu,0}^T(\{0\}), \end{aligned} \quad (3.5)$$

where we used $\Sigma_{\mu,0}^T = \Sigma_{\mu,0}^T(\{0\}) \delta_0$, which follows from (1.34) and [KIM, Eq. (2.53)], which also imply (3.2). \square

Theorem 3.2 (Large disorder). *Given $T \geq 0$ and $\mu \in \mathbb{R}$, there exist $\lambda_2 < \infty$ and a finite constant C independent of λ , such that*

$$\Sigma_{\mu,\lambda}^T(\mathbb{R}) \leq C\lambda^{-\frac{1}{4}} \quad \text{for all } \lambda \geq \lambda_2. \quad (3.6)$$

In particular, we have

$$\text{s-lim}_{\lambda \rightarrow \infty} \Sigma_{\mu,\lambda}^T = 0. \quad (3.7)$$

Proof. Fix $T \geq 0$ and $\mu \in \mathbb{R}$. Recall that there exists $\lambda_0 > 0$ such that $\Xi_0 = \mathbb{R}$ for $\lambda \geq \lambda_0$. Thus it follows from (1.22), proceeding as in [KIM, Eq. (3.19)], that for all $\lambda \geq \lambda_0$ we have (\hat{x}_1 denotes the unit vector in the x_1 -direction)

$$\begin{aligned} \Sigma_{\mu,\lambda}^T(\mathbb{R}) &= \pi \langle\langle \dot{X}_1, Y_{\mu,\lambda}^T \rangle\rangle = -\pi \mathbb{E} \left\{ \langle X_1^2 H_{\omega,\lambda} \delta_0, f_\mu^T(H_{\omega,\lambda}) \delta_0 \rangle \right\} \\ &= -\pi \mathbb{E} \left\{ \langle \delta_{\hat{x}_1} + \delta_{-\hat{x}_1}, f_\mu^T(H_{\omega,\lambda}) \delta_0 \rangle \right\} = -2\pi \text{Re } \mathbb{E} \left\{ \langle \delta_{\hat{x}_1}, f_\mu^T(H_{\omega,\lambda}) \delta_0 \rangle \right\}, \end{aligned} \quad (3.8)$$

where we used covariance for the last equality. In particular, we have

$$\Sigma_{\mu,\lambda}^T(\mathbb{R}) \leq 2\pi \left| \mathbb{E} \left\{ \langle \delta_{x_1}^{\wedge}, f_{\mu}^T(H_{\omega,\lambda}) \delta_0 \rangle \right\} \right|. \quad (3.9)$$

Let $\widetilde{H}_{\omega,\lambda} := -\lambda^{-1}\Delta + V_{\omega}$, so $H_{\omega,\lambda} = \lambda\widetilde{H}_{\omega,\lambda}$. Note that $\widetilde{\mathfrak{S}}_{\lambda} := \frac{1}{\lambda}\mathfrak{S}_{\lambda} = \sigma(\widetilde{H}_{\omega,\lambda})$ with probability one. We have

$$f_{\mu}^T(H_{\omega,\lambda}) = F_{\lambda}(\widetilde{H}_{\omega,\lambda}), \quad \text{where} \quad F_{\lambda} := f_{\frac{\mu}{\lambda}}^{\frac{T}{\lambda}}. \quad (3.10)$$

Without loss of generality we take $\lambda \geq \lambda_0$ large enough to ensure (recall(1.3)-(1.4))

$$\widetilde{\mathfrak{S}}_{\lambda} \subset [v_- - 1, v_+ + 1] \quad \text{and} \quad |\mu| \leq \lambda^{\frac{3}{4}} \quad (\text{i.e., } \frac{|\mu|}{\lambda} \leq \lambda^{-\frac{1}{4}}). \quad (3.11)$$

We fix a (λ -independent) function $h \in C_c^{\infty}(\mathbb{R})$, $0 \leq h \leq 1$, such that

$$\begin{aligned} h(s) &:= 1 \quad \text{if } s \in [v_- - 1, v_+ + 1], \quad h(s) := 0 \quad \text{if } s \notin [v_- - 3, v_+ + 3], \\ |h^{(r)}| &\leq \gamma \chi_{[v_- - 3, v_+ + 3] \setminus [v_- - 1, v_+ + 1]} \quad \text{for } r = 1, 2, 3 \end{aligned} \quad (3.12)$$

where $\gamma > 0$ is some universal constant. Note that

$$J_{\lambda}(\widetilde{H}_{\omega,\lambda}) = F_{\lambda}(\widetilde{H}_{\omega,\lambda}), \quad \text{where} \quad J_{\lambda} := hF_{\lambda}. \quad (3.13)$$

For each λ we fix an even function $g_{\lambda} \in C_c^{\infty}(\mathbb{R})$, $0 \leq g_{\lambda} \leq 1$, such that

$$\begin{aligned} g_{\lambda}(s) &:= 1 \quad \text{if } |s| \leq 3\lambda^{-\frac{1}{4}}, \quad g_{\lambda}(s) := 0 \quad \text{if } |s| \geq 5\lambda^{-\frac{1}{4}}, \\ |g_{\lambda}^{(r)}| &\leq \gamma \lambda^{\frac{r}{4}} \chi_{[-5\lambda^{-\frac{1}{4}}, 5\lambda^{-\frac{1}{4}}]} \quad \text{for } r = 1, 2, 3. \end{aligned} \quad (3.14)$$

We write

$$J_{\lambda} = J_{1,\lambda} + J_{2,\lambda}, \quad \text{where} \quad J_{1,\lambda} := g_{\lambda}J_{\lambda} \quad \text{and} \quad J_{2,\lambda} := (1 - g_{\lambda})J_{\lambda}. \quad (3.15)$$

Let $\widetilde{\mathcal{N}}_{\lambda}$ denote the density of states measure for $\widetilde{H}_{\omega,\lambda}$, i.e.,

$$\widetilde{\mathcal{N}}_{\lambda}(B) := \mathbb{E} \left\{ \langle \delta_0, \chi_B(\widetilde{H}_{\omega,\lambda}) \delta_0 \rangle \right\} \quad \text{for Borel sets } B \subset \mathbb{R}. \quad (3.16)$$

It follows from the Wegner estimate that $\widetilde{\mathcal{N}}_{\lambda}$ is absolutely continuous, and its Lebesgue density obeys the λ -independent bound $\frac{d\widetilde{\mathcal{N}}_{\lambda}}{dE} \leq \|\rho\|_{\infty}$. Thus, we get, using also $0 \leq J_{1,\lambda} \leq \chi_{[-5\lambda^{-\frac{1}{4}}, 5\lambda^{-\frac{1}{4}}]}$ (recall $0 \leq F_{\lambda} \leq 1$),

$$\begin{aligned} \left| \mathbb{E} \left\{ \langle \delta_{x_1}^{\wedge}, J_{1,\lambda}(\widetilde{H}_{\omega,\lambda}) \delta_0 \rangle \right\} \right| &\leq \mathbb{E} \left\{ \left\| \left(J_{1,\lambda}(\widetilde{H}_{\omega,\lambda}) \right)^{\frac{1}{2}} \delta_{x_1}^{\wedge} \right\| \left\| \left(J_{1,\lambda}(\widetilde{H}_{\omega,\lambda}) \right)^{\frac{1}{2}} \delta_0 \right\| \right\} \\ &\leq \mathbb{E} \left\{ \left\| \left(J_{1,\lambda}(\widetilde{H}_{\omega,\lambda}) \right)^{\frac{1}{2}} \delta_0 \right\|^2 \right\} = \mathbb{E} \left\{ \langle \delta_0, J_{1,\lambda}(\widetilde{H}_{\omega,\lambda}) \delta_0 \rangle \right\} \\ &= \int_{\mathbb{R}} d\widetilde{\mathcal{N}}_{\lambda}(E) J_{1,\lambda}(E) \leq \|\rho\|_{\infty} \int_{\mathbb{R}} dE J_{1,\lambda}(E) \leq 10 \|\rho\|_{\infty} \lambda^{-\frac{1}{4}}. \end{aligned} \quad (3.17)$$

We now estimate $\mathbb{E} \left\{ \langle \delta_{x_1}^{\wedge}, J_{2,\lambda}(\widetilde{H}_{\omega,\lambda}) \delta_0 \rangle \right\}$. Note that for any bounded measurable function k on \mathbb{R} we have

$$\langle \delta_{x_1}^{\wedge}, k(V_{\omega}) \delta_0 \rangle = \langle \delta_{x_1}^{\wedge}, \delta_0 \rangle = 0. \quad (3.18)$$

Thus, with probability one we have

$$\begin{aligned} \langle \delta_{x_1}^\wedge, J_{2,\lambda}(\widetilde{H}_{\omega,\lambda})\delta_0 \rangle &= \langle \delta_{x_1}^\wedge, (J_{2,\lambda}(\widetilde{H}_{\omega,\lambda}) - J_{2,\lambda}(V_\omega))\delta_0 \rangle \\ &= \frac{1}{\lambda} \int_{\mathbb{C}} d\widetilde{J_{2,\lambda}}(z) \langle \delta_{x_1}^\wedge, \frac{1}{\widetilde{H}_{\omega,\lambda} - z} \Delta \frac{1}{V_\omega - z} \delta_0 \rangle, \end{aligned} \quad (3.19)$$

where the second equality relies on the Helffer-Sjöstrand formula. (We refer to [HS, App. B] for a review of the Helffer-Sjöstrand formula.) Given a smooth function ζ on the real line, $\widetilde{\zeta}$ denotes an almost analytic extension of ζ to the complex plane. We recall the estimate

$$\int_{\mathbb{C}} |d\widetilde{\zeta}(z)| \frac{1}{|\operatorname{Im} z|^2} \leq c_3 \{ \{ \zeta \} \}_3, \quad (3.20)$$

where c_3 is a finite constant independent of the function ζ , and

$$\{ \{ \zeta \} \}_3 := \sum_{r=0}^3 \int_{\mathbb{R}} ds |\zeta^{(r)}(s)| (1 + |s|^2)^{\frac{r-1}{2}}. \quad (3.21)$$

We thus conclude that

$$| \langle \delta_{x_1}^\wedge, J_{2,\lambda}(\widetilde{H}_{\omega,\lambda})\delta_0 \rangle | \leq \frac{2dc_3}{\lambda} \{ \{ J_{2,\lambda} \} \}_3, \quad (3.22)$$

and need to estimate $\{ \{ J_{2,\lambda} \} \}_3$. If $T = 0$ we have $F'_\lambda(E) = 0$ for $|E| \geq 3\lambda^{-\frac{1}{4}}$. If $T > 0$ we have

$$\begin{aligned} |F'_\lambda(E)| &= \frac{\lambda}{2T} \operatorname{sech}^2 \left(\frac{\lambda}{2T} (E - \frac{\mu}{\lambda}) \right) \leq \frac{\lambda}{2T} \operatorname{sech}^2 \left(\frac{\lambda}{2T} (2\lambda^{-\frac{1}{4}}) \right) \\ &= \frac{\lambda}{2T} \operatorname{sech}^2 \left(\frac{\lambda^{\frac{3}{4}}}{T} \right) \quad \text{for } |E| \geq 3\lambda^{-\frac{1}{4}}, \end{aligned} \quad (3.23)$$

with similar estimates for $|F_\lambda^{(r)}(E)|$, $r = 2, 3$. (We also used (3.11).) We conclude that there is a constant $K > 0$ such that

$$\max_{r=1,2,3} |F_\lambda^{(r)}(E)| \leq K < \infty \quad \text{for } |E| \geq 3\lambda^{-\frac{1}{4}}. \quad (3.24)$$

Combining with (3.11), (3.12), and (3.14) we conclude that

$$\{ \{ J_{2,\lambda} \} \}_3 \leq K' \lambda^{\frac{3}{4}}, \quad \text{where } K' \text{ is a finite constant independent of } \lambda. \quad (3.25)$$

Combining (3.22) and (3.25) we get the deterministic bound

$$| \langle \delta_{x_1}^\wedge, J_{2,\lambda}(\widetilde{H}_{\omega,\lambda})\delta_0 \rangle | \leq 2dc_3 K' \lambda^{-\frac{1}{4}}, \quad (3.26)$$

valid on an event of probability one.

Taking the expectation of (3.26), and using (3.9), (3.13) and (3.17), we get

$$\Sigma_{\mu,\lambda}^T(\mathbb{R}) \leq C \lambda^{-\frac{1}{4}} \quad \text{for all } \lambda \geq \lambda_2, \quad (3.27)$$

where $\lambda_2 < \infty$ and C is finite constant independent of λ . \square

4. ELECTROMAGNETIC ENERGY ABSORPTION

We now consider the Anderson model as in (1.1) and an electric field as in (1.6) such that $\mathcal{E} \in L^1(\mathbb{R})$. Notice that this is equivalent to assume

$$\mathcal{E}(t) = \int_{\mathbb{R}} d\nu e^{i\nu t} \widehat{\mathcal{E}}(\nu), \quad \text{where } \mathcal{E}, \widehat{\mathcal{E}} \in L^1(\mathbb{R}) \text{ with } \widehat{\mathcal{E}}(\nu) = \overline{\widehat{\mathcal{E}}(-\nu)}. \quad (4.1)$$

Note that (4.1) implies $\mathcal{E}, \widehat{\mathcal{E}} \in C(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

With the extra assumption of $\mathcal{E} \in L^1(\mathbb{R})$ we can proceed without adiabatic switching, i.e., with $\eta = 0$. If $T > 0$ or $\mu \in \Xi_0$, the energy of the system at time t is given by $\mathcal{T}(H_\omega(t) \varrho_{\mu,\omega}^T(t))$. The total energy the solid absorbs during all times from the electric field is given by

$$\begin{aligned} W_\mu^T(\mathcal{E}) &= \lim_{t \rightarrow \infty} \mathcal{T}(H_\omega(t) \varrho_{\mu,\omega}^T(t)) - \lim_{t \rightarrow -\infty} \mathcal{T}(H_\omega(t) \varrho_{\mu,\omega}^T(t)) \\ &= \int_{\mathbb{R}} dt \frac{d}{dt} \mathcal{T}(H_\omega(t) \varrho_{\mu,\omega}^T(t)) \end{aligned} \quad (4.2)$$

$$\begin{aligned} &= \int_{\mathbb{R}} dt \left(\mathcal{T}(-i[H_\omega(t), \varrho_{\mu,\omega}^T(t)] H_\omega(t)) + \mathcal{T}(\varrho_{\mu,\omega}^T(t) \frac{d}{dt} H_\omega(t)) \right) \\ &= i \int_{\mathbb{R}} dt \mathcal{E}(t) \mathcal{T}(\varrho_{\mu,\omega}^T(t) [X_1, H_\omega(t)]) = \int_{\mathbb{R}} dt \mathcal{E}(t) J(t, \mu, T, \mathcal{E}). \end{aligned} \quad (4.3)$$

Here we used (1.8), the cyclic invariance of \mathcal{T} to get

$$\mathcal{T}(-i[H_\omega(t), \varrho_{\mu,\omega}^T(t)] H_\omega(t)) = 0, \quad (4.4)$$

as well as (1.7), (1.10) and (1.9). Note that $J(t, \mu, T, \mathcal{E})$, and hence $W_\mu^T(\mathcal{E})$, are real-valued since $H_\omega(t)$ is self-adjoint and $\varrho_{\mu,\omega}^T(t) \geq 0$.

The absorption of electromagnetic energy in linear response theory can now be seen to be well defined in terms of the linear-response current $J_{\text{lin}}(t; \mu, T, \mathcal{E})$ (see (1.11) and (1.23)):

$$\begin{aligned} W_{\mu,\text{lin}}^T(\mathcal{E}) &:= \lim_{\alpha \rightarrow 0} \frac{W_\mu^T(\alpha \mathcal{E})}{\alpha^2} = \int_{\mathbb{R}} dt \mathcal{E}(t) J_{\text{lin}}(t; \mu, T, \mathcal{E}) \\ &= \frac{1}{\pi} \int_{\mathbb{R}} dt \mathcal{E}(t) \int_{-\infty}^t ds \mathcal{E}(s) \int_{\mathbb{R}} \Sigma_\mu^T(d\nu) e^{-i(t-s)\nu} \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} dt ds \mathcal{E}(t) \mathcal{E}(s) \chi_{[0,\infty[}(t-s) \int_{\mathbb{R}} \Sigma_\mu^T(d\nu) e^{-i(t-s)\nu} \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} dt ds \mathcal{E}(t) \mathcal{E}(s) \chi_{[0,\infty[}(t-s) \int_{\mathbb{R}} \Sigma_\mu^T(d\nu) \cos((t-s)\nu) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} dt ds \mathcal{E}(t) \mathcal{E}(s) \int_{\mathbb{R}} \Sigma_\mu^T(d\nu) \cos((t-s)\nu) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} dt ds \mathcal{E}(t) \mathcal{E}(s) \int_{\mathbb{R}} \Sigma_\mu^T(d\nu) e^{-i(t-s)\nu} \\ &= 2\pi \int_{\mathbb{R}} \Sigma_\mu^T(d\nu) |\widehat{\mathcal{E}}(\nu)|^2, \end{aligned} \quad (4.5)$$

where we used the fact that Σ_μ^T is an even measure.

Thus we proved the following theorem. (See [BrPK4, Theorem 4.7] for an analogous result in the BPK setting.)

Theorem 4.1 (Electromagnetic energy absorption). *Consider the Anderson model as in (1.1) and an electric field \mathcal{E} as in (4.1). Suppose either $T > 0$ or $\mu \in \Xi_0$. Then*

$$W_{\mu,\text{lin}}^T(\mathcal{E}) = 2\pi \int_{\mathbb{R}} \Sigma_{\mu}^T(d\nu) |\widehat{\mathcal{E}}(\nu)|^2 \geq 0. \quad (4.6)$$

In particular, we have

$$W_{\mu,\text{lin}}^T(\mathcal{E}) = 2\pi \int_{\mathbb{R}} \Gamma_{\mu}^T(d\nu) |\widehat{\mathcal{E}}(\nu)|^2 \geq 0 \quad \text{if} \quad \widehat{\mathcal{E}}(0) = 2\pi \int_{\mathbb{R}} dt \mathcal{E}(t) = 0. \quad (4.7)$$

In addition, for all $T > 0$ and $\mu \in \mathbb{R}$ we have

$$W_{\mu,\text{lin}}^T(\mathcal{E}) \geq 2\pi \int_{\mathbb{R}} \Gamma_{\mu}^T(d\nu) |\widehat{\mathcal{E}}(\nu)|^2 \geq \frac{\pi^2}{2T} C_{\mu}^T \int_{\mathbb{R}} \Upsilon(d\nu) |\widehat{\mathcal{E}}(\nu)|^2. \quad (4.8)$$

Remark 4.2. *It follows from Remark 2.2 that only frequencies $\nu \in \sigma(\mathcal{L})$ contribute to $W_{\mu,\text{lin}}^T(\mathcal{E})$. In particular, we have*

$$W_{\mu,\text{lin}}^T(\mathcal{E}) = 2\pi \int_{\sigma(\mathcal{L})} \Sigma_{\mu}^T(d\nu) |\widehat{\mathcal{E}}(\nu)|^2 \quad \text{if either} \quad T > 0 \quad \text{or} \quad \mu \in \Xi_0. \quad (4.9)$$

REFERENCES

- [AG] Aizenman, M., Graf, G.M.: Localization bounds for an electron gas. J. Phys. A: Math. Gen. **31**, 6783–6806, (1998)
- [BES] Bellissard, J., van Elst, A., Schulz-Baldes, H.: The noncommutative geometry of the quantum Hall effect. J. Math. Phys. **35**, 5373–5451 (1994)
- [BeP] Benderskij, M.M., Pastur, L.A.: On the spectrum of the one-dimensional Schrödinger equation with a random potential. Math. USSR Sb. **11**, 245–256 (1970) [Russian original: Mat. Sb. **82**, 273–284 (1970)]
- [BoGKS] Bouclet, J.M., Germinet, F., Klein, A., Schenker, J.: Linear response theory for magnetic Schrödinger operators in disordered media. J. Funct. Anal. **226**, 301–372 (2005). doi:10.1016/j.jfa.2005.02.002
- [BrPK1] Bru, J.-B., de Siqueira Pedra, W., Kurig, C.: Heat production of non-interacting Fermions subjected to electric fields in disordered media. Preprint mp_arc 13-29. To appear in Commun. Pure Appl. Math.
- [BrPK2] Bru, J.-B., de Siqueira Pedra, W., Kurig, C.: Microscopic conductivity distributions of non-interacting Fermions. Preprint mp_arc 13-68
- [BrPK3] Bru, J.-B., Pedra, de Siqueira Pedra, W., Kurig, C.: AC conductivity measure from heat production of free Fermions in disordered media. Preprint mp_arc 13-88
- [BrPK4] Bru, J.-B., de Siqueira Pedra, W., Kurig, C.: Macroscopic conductivity of free Fermions in disordered media. Preprint mp_arc 13-93
- [CL] Carmona, R., Lacroix, J.: *Spectral theory of random Schrödinger operators*. Birkhäuser, Boston, 1990
- [GK] Germinet, F., Klein, A.: New characterizations of the region of complete localization for random Schrödinger operators. J. Stat. Phys. **122**, 73–94 (2006). doi:10.1007/s10955-005-8068-9
- [HS] Hunziker, W., Sigal, I.M.: Time-dependent scattering theory of N -body quantum systems. Rev. Math. Phys. **12**, 1033–1084 (2000)
- [KP] Khorunzhy, A.M., Pastur, L.A.: Limits of infinite interaction radius, dimensionality and the number of components for random operators with off-diagonal randomness. Commun. Math. Phys. **153**, 605–646 (1993)
- [Ki] Kirsch, W.: An invitation to random Schrödinger operators. In: Panor. Synthèses **25**. *Random Schrödinger Operators*. Soc. Math. France, Paris, 2008, pp. 1–119
- [KiLP] Kirsch, W., Lenoble, O., Pastur, L.A.: On the Mott formula for the ac conductivity and binary correlators in the strong localization regime of disordered systems. J. Phys. A **36**, 12157–12180 (2003)

- [KiM] Kirsch, W., Martinelli, F.: On the ergodic properties of the spectrum of general random operators. *J. Reine Angew. Math.* **334**, 141–156 (1982)
- [KILM] Klein, A., Lenoble, O., Müller, P.: On Mott’s formula for the ac-conductivity in the Anderson model. *Ann. Math. (2)* **166**, 549–577 (2007). <http://annals.math.princeton.edu/annals/2007/166-2/p07.xhtml>
- [KIM] Klein, A., Müller, P.: The conductivity measure for the Anderson model. *Zh. Mat. Fiz. Anal. Geom.* **4**, 128–150 (2008). <http://jmage.ilt.kharkov.ua/list.php>
- [LGP] Lifshits, I.M., Gredeskul, A.G., Pastur, L.A.: *Introduction to the Theory of Disordered Systems*. Wiley, New York, 1988
- [P1] Pastur, L.A.: On the Schrödinger equation with a random potential. *Theor. Math. Phys.* **6**, 299–306 (1971) [Russian original: *Teor. Mat. Fiz.* **6**, 415–424 (1971)]
- [P2] Pastur, L.A.: Spectra of random self adjoint operators. *Russ. Math. Surveys* **28**, 1–67 (1973) [Russian original: *Usp. Mat. Nauk* **28**, 3–64 (1973)]
- [P3] Pastur, L.A.: On the spectrum of random Jacobi matrices and the Schrödinger equation on the whole axis with random potential. Preprint, Kharkov, 1974
- [P4] Pastur, L.A.: Spectral properties of disordered systems in one-body approximation. *Commun. Math. Phys.* **75**, 179–196 (1980)
- [P5] Pastur, L.A.: Spectral properties of random selfadjoint operators and matrices (a survey). *Proc. St. Petersburg Math. Soc.*, Vol. IV, 153–195, Amer. Math. Soc. Transl. Ser. 2, 188, Providence, RI, 1999
- [P6] Pastur, L.A.: On some asymptotic formulas in the strong localization regime of the theory of disordered systems. *Oper. Theory Adv. Appl.*, vol. 108, Birkhäuser, Basel, 1999, pp. 129–148
- [PF] Pastur, L., Figotin, A.: *Spectra of Random and Almost-Periodic Operators*. Springer, Heidelberg, 1992

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